# Typical rank of $m \times n \times (m-1)n$ tensors with $3 \le m \le n$ over the real number field

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**Abstract**

Tensor type data are used recently in various application fields, and then a typical rank is important. Let $3 \le m \le n$. We study typical ranks of $m \times n \times (m-1)n$ tensors over the real number field. Let $\rho$ be the Hurwitz-Radon function defined as $\rho(n) = 2^b + 8c$ for nonnegative integers $a, b, c$ such that $n = (2a+1)2^{b+4c}$ and $0 \le b < 4$. If $m \le \rho(n)$, then the set of $m \times n \times (m-1)n$ tensors has two typical ranks $(m-1)n, (m-1)n+1$. In this paper, we show that the converse is also true: if $m > \rho(n)$, then the set of $m \times n \times (m-1)n$ tensors has only one typical rank $(m-1)n$.

## 1 Introduction

An analysis of high dimensional arrays is getting frequently used. Kolda and Bader [6] introduced many applications of tensor decomposition analysis in various fields such as signal processing, computer vision, data mining, and others.

In this paper we concentrate to discuss 3-way arrays. A 3-way array

$$(a_{ijk})_{1 \le i \le m,\ 1 \le j \le n,\ 1 \le k \le p}$$

with size $(m, n, p)$ is called an $m \times n \times p$ tensor. A rank of a tensor $T$, denoted by $\operatorname{rank} T$, is defined as the minimal number of rank one tensors which describe $T$ as a sum. The rank depends on the base field. For example there is a $2 \times 2 \times 2$ tensor over the real number field whose rank is 3 but is 2 as a tensor over the complex number field.

Throughout this paper, we assume that the base field is the real number field $\mathbb{R}$. Let $\mathbb{R}^{m \times n \times p}$ be the set of $m \times n \times p$ tensors with Euclidean topology. A number $r$ is a typical rank of $m \times n \times p$ tensors if the set of tensors with rank $r$ contains a nonempty open semi-algebraic set of $\mathbb{R}^{m \times n \times p}$ (see Theorem 2.2). We denote by $\operatorname{typical\_rank}_{\mathbb{R}}(m, n, p)$ the set of typical ranks of $\mathbb{R}^{m \times n \times p}$. If $s$ (resp. $t$) is the minimal (resp. maximal) number of $\operatorname{typical\_rank}_{\mathbb{R}}(m, n, p)$, then

$$\operatorname{typical\_rank}_{\mathbb{R}}(m, n, p) = [s, t],$$

the interval of all integers between $s$ and $t$, including both, and $s$ is equal to the generic rank of the set of $m \times n \times p$ tensors over the complex number field [4]. In the case where $m = 2$, the set of typical ranks of $2 \times n \times p$ tensor is well-known [12]:

$$\text{typical\_rank}_{\mathbb{R}}(2,n,p) = \begin{cases} \{p\}, & n < p \le 2n \\ \{2n\}, & 2n < p \\ \{p, p+1\}, & n = p \ge 2 \end{cases}$$

Suppose that $3 \le m \le n$. If $p > (m-1)n$ then the set of typical ranks of $m \times n \times p$ tensors is just $\{\min(p, mn)\}$. If $p = (m-1)n$ then the set of typical ranks of $m \times n \times p$ tensor is $\{p\}$ or $\{p, p+1\}$ [11]. Until our paper [10], only a few cases where $\text{typical\_rank}_{\mathbb{R}}(m,n,(m-1)n) = \{(m-1)n, (m-1)n+1\}$ [2, 4] are known and we constructed infinitely many examples by using the concept of absolutely nonsingular tensors in [10]: If $m \le \rho(n)$ then $\text{typical\_rank}_{\mathbb{R}}(m,n,p) = \{p, p+1\}$, where $\rho(n)$ is the Hurwitz-Radon number given by $\rho(n) = 2^b + 8c$ for nonnegative integers $a, b, c$ such that $n = (2a+1)2^{b+4c}$ and $0 \le b < 4$.

The purpose of this paper is to completely determine the set of typical ranks of $m \times n \times (m-1)n$ tensors:

**Theorem 1.1** *Let $3 \le m \le n$ and $p = (m-1)n$. Then it holds*

$$\text{typical\_rank}_{\mathbb{R}}(m,n,p) = \begin{cases} \{p\}, & m > \rho(n) \\ \{p, p+1\}, & m \le \rho(n). \end{cases}$$

We denote an $m_1 \times m_2 \times m_3$ tensor $(x_{ijk})$ by $(X_1; \dots; X_{m_3})$, where $X_t = (x_{ijt})$ is an $m_1 \times m_2$ matrix for each $1 \le t \le m_3$. Let $3 \le m \le n$ and $p = (m-1)n$. For an $n \times p \times m$ tensor $X = (X_1; \dots; X_{m-1}; X_m)$, let $H(X)$ and $\hat{H}(X)$ be a $p \times p$ matrix and an $mn \times p$ matrix respectively defined as follows.

$$H(X) = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{m-1} \end{pmatrix}, \quad \hat{H}(X) = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{pmatrix}$$

Let

$$\mathfrak{R} = \{X \in \mathbb{R}^{n \times p \times m} \mid H(X) \text{ is nonsingular}\}.$$

This is a nonempty Zariski open set. For $X = (X_1; \dots; X_{m-1}; X_m) \in \mathfrak{R}$, we see

$$\hat{H}(X)H(X)^{-1} = \begin{pmatrix} E_n & & & \\ & E_n & & \\ & & \ddots & \\ & & & E_n \\ Y_1 & Y_2 & \cdots & Y_{m-1} \end{pmatrix},$$

where $(Y_1, Y_2, \dots, Y_{m-1}) = X_m H(X)^{-1}$. Note that $\operatorname{rank} X \ge p$ for $X \in \mathfrak{R}$. Let $h$ be an isomorphism from the set of $n \times p$ matrices to $\mathbb{R}^{n \times n \times (m-1)}$ given by

$$(Y_1, Y_2, \dots, Y_{m-1}) \mapsto (Y_1; Y_2; \dots; Y_{m-1}).$$

Then $h(X_mH(X)^{-1}) \in \mathbb{R}^{n\times n\times(m-1)}$. We consider the following subsets of $\mathbb{R}^{n\times n\times(m-1)}$. For $Y = (Y_1;Y_2;\dots;Y_{m-1}) \in \mathbb{R}^{n\times n\times(m-1)}$ and $\boldsymbol{a} = (a_1,\dots,a_{m-1},a_m)^\top \in \mathbb{R}^m$, let

$$M(\boldsymbol{a},Y) = \sum_{k=1}^{m-1} a_k Y_k - a_m E_n$$

and set

$$\mathfrak{C} = \{Y \in \mathbb{R}^{n\times n\times(m-1)} \mid |M(\boldsymbol{a},Y)| < 0 \text{ for some } \boldsymbol{a} \in \mathbb{R}^m\}$$

and

$$\mathfrak{A} = \{Y \in \mathbb{R}^{n\times n\times(m-1)} \mid |M(\boldsymbol{a},Y)| > 0 \text{ for all } \boldsymbol{a} \neq \boldsymbol{0}\}.$$

The subsets $\mathfrak{C}$ and $\mathfrak{A}$ are open sets in Euclidean topology and $\overline{\mathfrak{C}} \cup \overline{\mathfrak{A}} = \mathbb{R}^{n\times n\times(m-1)}$. In [10], we show that $\mathfrak{A}$ is not empty if and only if $m \le \rho(n)$ and that $\operatorname{rank} X > p$ for any $X \in \mathfrak{R}$ with $h(X_mH(X)^{-1}) \in \mathfrak{A}$. In this paper, we show that there exists an open subset $\mathfrak{F}$ of $\mathfrak{C}$ such that $\overline{\mathfrak{F}} = \overline{\mathfrak{C}}$ and $\operatorname{rank} X = p$ for any $X \in \mathfrak{R}$ with $h(X_mH(X)^{-1}) \in \mathfrak{F}$.

# 2 Typical rank

Due to [8, 11] and others, a number $r$ is a typical rank of tensors of $\mathbb{R}^{m_1\times m_2\times m_3}$ if the subset of tensors of $\mathbb{R}^{m_1\times m_2\times m_3}$ of rank $r$ has nonzero volume. In this paper, we adopt the algebraic definition due to Friedland. These definitions are equivalent, since for any $r \ge 0$, the set of tensors of rank $r$ is a semi-algebraic set by the Tarski-Seidenberg principle (cf. [1]).

For $\boldsymbol{x} = (x_1,\dots,x_{m_1})^\top \in \mathbb{C}^{m_1}$, $\boldsymbol{y} = (y_1,\dots,y_{m_2})^\top \in \mathbb{C}^{m_2}$, and $\boldsymbol{z} = (z_1,\dots,z_{m_3})^\top \in \mathbb{C}^{m_3}$, we denote $(x_iy_jz_k) \in \mathbb{C}^{m_1\times m_2\times m_3}$ by $\boldsymbol{x}\otimes\boldsymbol{y}\otimes\boldsymbol{z}$. Let $f_t\colon (\mathbb{C}^{m_1}\times\mathbb{C}^{m_2}\times\mathbb{C}^{m_3})^t \to \mathbb{C}^{m_1\times m_2\times m_3}$ be a map given by

$$f_t(\boldsymbol{x}_{1,1},\boldsymbol{x}_{1,2},\boldsymbol{x}_{1,3},\dots,\boldsymbol{x}_{t,1},\boldsymbol{x}_{t,2},\boldsymbol{x}_{t,3}) = \sum_{\ell=1}^{t} \boldsymbol{x}_{\ell,1}\otimes\boldsymbol{x}_{\ell,2}\otimes\boldsymbol{x}_{\ell,3}.$$

Let $S$ be a subset of $\mathbb{R}^{m_1\times m_2\times m_3}$. $S$ is called semi-algebraic if it is a finite Boolean combination (that is, a finite composition of disjunctions, conjunctions and negatios) of sets of the form

$$\{(a_{ijk}) \in \mathbb{R}^{m_1\times m_2\times m_3} \mid f(a_{111},\dots,a_{m_1,m_2,m_3}) > 0\} \tag{2.1}$$

and

$$\{(a_{ijk}) \in \mathbb{R}^{m_1\times m_2\times m_3} \mid g(a_{111},\dots,a_{m_1,m_2,m_3}) = 0\},$$

where $f$ and $g$ are polynomials in $m_1m_2m_3$ indeterminates $x_{111},\dots,x_{m_1,m_2,m_3}$ over $\mathbb{R}$. Then $S$ is an open semi-algebraic set if and only if it is expressed as a finite Boolean combinations of sets of the form (2.1), and it is a dense open semi-albebraic set if and only if it is a Zariski open set, that is, expressed as

$$\{(a_{ijk}) \in \mathbb{R}^{m_1\times m_2\times m_3} \mid g(a_{111},\dots,a_{m_1,m_2,m_3}) \neq 0\}.$$

**Theorem 2.2 ([4, Theorem 7.1])** *The space $\mathbb{R}^{m_1\times m_2\times m_3}$, $m_1,m_2,m_3\in\mathbb{N}$, contains a finite number of open connected disjoint semi-algebraic sets $O_1,\dots,O_M$ satisfying the following properties.*

(1) $\mathbb{R}^{m_1\times m_2\times m_3}\setminus\cup_{i=1}^M O_i$ *is a closed semi-algebraic set* $\mathbb{R}^{m_1\times m_2\times m_3}$ *of dimension less than* $m_1m_2m_3$.

(2) *Each* $T\in O_i$ *has rank* $r_i$ *for* $i=1,\dots,M$.

(3) *The number* $\min(r_1,\dots,r_M)$ *is equal to the generic rank* $\operatorname{grank}(m_1,m_2,m_3)$ *of* $\mathbb{C}^{m_1\times m_2\times m_3}$, *that is, the minimal* $t\in\mathbb{N}$ *such that the closure of the image of* $f_t$ *is equal to* $\mathbb{C}^{m_1\times m_2\times m_3}$.

(4) $\operatorname{mtrank}(m_1,m_2,m_3):=\max(r_1,\dots,r_M)$ *is the minimal* $t\in\mathbb{N}$ *such that the closure of* $f_t((\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}\times\mathbb{R}^{m_3})^k)$ *is equal to* $\mathbb{R}^{m_1\times m_2\times m_3}$.

(5) *For each integer* $r\in[\operatorname{grank}(m_1,m_2,m_3),\operatorname{mtrank}(m_1,m_2,m_3)]$, *there exists* $r_i=r$ *for some integer* $i\in[1,M]$.

**Definition 2.3** A positive number $r$ is called a typical rank of $\mathbb{R}^{m_1\times m_2\times m_3}$ if

$$r\in[\operatorname{grank}(m_1,m_2,m_3),\operatorname{mtrank}(m_1,m_2,m_3)].$$

Put

$$\text{typical\_rank}_{\mathbb{R}}(m_1,m_2,m_3)=[\operatorname{grank}(m_1,m_2,m_3),\operatorname{mtrank}(m_1,m_2,m_3)].$$

We state basic facts.

**Proposition 2.4** *Let $r$ be a positive number and $U$ a nonempty open set of $\mathbb{R}^{m_1\times m_2\times m_3}$. If every tensor of $U$ has rank $r$, then $r$ is a typical rank of $\mathbb{R}^{m_1\times m_2\times m_3}$.*

**Proof** Let $O_1,\dots,O_M$ be open connected disjoint semi-algebraic sets as in Theorem 2.2. Since $\dim(\mathbb{R}^{m_1\times m_2\times m_3}\setminus\cup_{i=1}^M O_i)<m_1m_2m_3$, there exists $i\in[1,M]$ such that $U\cap O_i$ is not empty.

**Proposition 2.5** *Let $m_1,m_2,m_3,m_4\in\mathbb{N}$ with $m_3<m_4$. Then*

$$\operatorname{grank}(m_1,m_2,m_3)\le\operatorname{grank}(m_1,m_2,m_4)$$

*and*

$$\operatorname{mtrank}(m_1,m_2,m_3)\le\operatorname{mtrank}(m_1,m_2,m_4).$$

**Proof** Let $U$ be the nonempty Zariski open subset $U$ of $\mathbb{C}^{m_1\times m_2\times m_4}$ consisting of all tensors of rank $\operatorname{grank}(m_1,m_2,m_4)$ and put

$$V=\{(Y_1;Y_2;\dots;Y_{m_3})\in\mathbb{C}^{m_1\times m_2\times m_3}\mid(Y_1;Y_2;\dots;Y_{m_4})\in U\}.$$

Then $V$ is a nonempty Zariski open set of $\mathbb{C}^{m_1\times m_2\times m_3}$. For the subset $U'$ of $\mathbb{C}^{m_1\times m_2\times m_3}$ consisting of all tensors of rank $\operatorname{grank}(m_1,m_2,m_3)$, the intersection $V\cap U'$ is a nonempty

Zariski open set. Since $\operatorname{rank} Y \leq \operatorname{rank}(Y;X)$ for $Y \in \mathbb{C}^{m_1\times m_2\times m_3}$ and $(Y;X) \in \mathbb{C}^{m_1\times m_2\times m_4}$, we see

$$\operatorname{grank}(m_1,m_2,m_3) \leq \operatorname{grank}(m_1,m_2,m_4).$$

Next, take an open semi-algebraic set $V$ of $\mathbb{R}^{m_1\times m_2\times m_3}$ consisting of tensors of rank $\operatorname{mtrank}(m_1,m_2,m_3)$. Then there are $s \in \operatorname{typical\_rank}_{\mathbb{R}}(m_1,m_2,m_4)$ and an open semi-algebraic set $O$ of $\mathbb{R}^{m_1\times m_2\times m_4}$ consisting of tensors of rank $s$ such that $\{(A;B) | A \in V, B \in \mathbb{R}^{m_1\times m_2\times (m_4-m_3)}\} \cap O \neq \varnothing$. Thus

$$\operatorname{mtrank}(m_1,m_2,m_3) \leq s \leq \operatorname{mtrank}(m_1,m_2,m_4).$$

∎

The action of $\mathrm{GL}(m)\times\mathrm{GL}(n)\times\mathrm{GL}(p)$ on $\mathbb{R}^{m\times n\times p}$ is given as follows. Let $P=(p_{ij}) \in \mathrm{GL}(n)$, $Q=(q_{ij}) \in \mathrm{GL}(m)$, and $R=(r_{ij}) \in \mathrm{GL}(p)$. The tensor $(b_{ijk}) = (P,Q,R)\cdot(a_{ijk})$ is defined as

$$b_{ijk} = \sum_{s=1}^{m}\sum_{t=1}^{n}\sum_{u=1}^{p} p_{is}q_{jt}r_{ku}a_{stu}.$$

Therefore,

$$(P,Q,R)\cdot(A_1;\ldots;A_p) = (\sum_{u=1}^{p} r_{1u}PA_uQ^\top;\ldots;\sum_{u=1}^{p} r_{pu}PA_uQ^\top).$$

**Definition 2.6** Two tensors $A$ and $B$ is called *equivalent* if there exists $g \in \mathrm{GL}(m)\times\mathrm{GL}(n)\times\mathrm{GL}(p)$ such that $B = g\cdot A$.

**Proposition 2.7** *If two tensors are equivalent, then they have the same rank.*

A $1\times m_2\times m_3$ tensor $T$ is an $m_2\times m_3$ matrix and $\operatorname{rank} T$ is equal to the matrix rank. The following three propositions are well-known.

**Proposition 2.8** *Let $m_1,m_2,m_3 \in \mathbb{N}$ with $2 \leq m_1 \leq m_2 \leq m_3$. If $m_1m_2 \leq m_3$, then typical rank of $\mathbb{R}^{m_1\times m_2\times m_3}$ is only one integer $m_1m_2$.*

**Proposition 2.9** *An $m_1\times m_2\times m_3$ tensor $(Y_1;\ldots;Y_{m_3})$ has rank less than or equal to $r$ if and only if there are an $m_1\times r$ matrix $P$, an $r\times m_2$ matrix $Q$, and $r\times r$ diagonal matrices $D_1,\ldots,D_{m_3}$ such that $Y_k = PD_kQ$ for $1 \leq k \leq m_3$.*

**Proposition 2.10** *Let $X=(x_{ijk})$ be an $m_1\times m_2\times m_3$ tensor. For an $m_2\times m_1\times m_3$ tensor $Y=(x_{jik})$ and an $m_1\times m_3\times m_2$ tensor $Z=(x_{ikj})$, it holds that*

$$\operatorname{rank} X = \operatorname{rank} Y = \operatorname{rank} Z.$$

For an integer $2 \leq m < n < 2m$, the number $n$ is an only typical rank of $\mathbb{R}^{m\times n\times 2}$. Indeed, it is known that

**Theorem 2.11 ([7])** *Let $2 \leq m < n$. There is an open dense semi-algebraic set $O$ of $\mathbb{R}^{m\times n\times 2}$ of which any tensor is equivalent to $((E_m,O);(O,E_m))$ which has rank $\min(n,2m)$.*

Furthermore, by Proposition 2.5, typical_rank$_{\mathbb{R}}(m,m,2)$ is equal to either $\{m\}$ or $\{m,m+1\}$. Let $U$ be an open subset of $\mathbb{R}^{m\times m\times 2}$ consisting of $(A;B)$ such that $A$ is an $m\times m$ nonsingular matrix and all eigenvalues of $A^{-1}B$ are distinct and contain non-real numbers. For $m\geq 2$, the set $U$ is not empty and any tensor of $U$ has rank $m+1$ (cf. [9, Theorem 4.6]) and therefore typical_rank$_{\mathbb{R}}(m,m,2)=\{m,m+1\}$ by Proposition 2.4.

**Theorem 2.12 ([11, Result 2])** *Let $m,n,\ell\in\mathbb{N}$ with $3\leq m\leq n\leq u$. If $(m-1)n<u<mn$, then typical rank of $\mathbb{R}^{m\times n\times u}$ is only one integer $u$.*

Ten Berge showed it by applying Fisher's result [3, Theorem 5.A.2] for a map defined by using the Moore-Penrose inverse. However the Moore-Penrose inverse is not continuous on the set of matrices and thus not analytic. So, until this section, we give another proof for reader's convenience.

Let $3\leq m\leq n$, $p=(m-1)n$, $p<u<mn$ and $q=u-p-1$. For $W\in M(n-1,n;\mathbb{R})$, the set of $(n-1)\times n$ matrices, we define a vector $W^{\perp}=(a_1,\dots,a_n)^{\top}$ in $\mathbb{R}^n$ by

$$a_j=(-1)^{n+j}|W_{[j]}|$$

for $j=1,\dots,n$, where $W_{[j]}$ is an $(n-1)\times(n-1)$ matrix obtained from $W$ by removing the $j$-th column.

The following properties are easily shown.

(1) $W^{\perp}=\mathbf{0}$ if and only if $\operatorname{rank}W<n-1$.

(2) $WW^{\perp}=\mathbf{0}$.

Let $A_k$ be an $n\times u$ matrix for $1\leq k\leq m$. Let $B_j$ be a $q\times u$ matrix defined by $(O_{p+1},E_q)$ for $j\leq p+1$, and by $(O_p,\boldsymbol{e}_{j-p-1},\mathrm{Diag}(E_{j-p-2},0,E_{u-j}))$ for $p+2\leq j\leq u$, where $E_k$ is the $k\times k$ identity matrix and $\boldsymbol{e}_j$ is the $j$-th column of the identity matrix with suitable size. Put

$$X_j=\begin{pmatrix}A_2-jA_1\\A_3-j^2A_1\\\vdots\\A_m-j^{m-1}A_1\end{pmatrix}\text{ and }Y_j=\begin{pmatrix}X_j\\B_j\end{pmatrix}\tag{2.13}$$

for $1\leq j\leq u$, and

$$H=(Y_1^{\perp},\dots,Y_u^{\perp}).\tag{2.14}$$

We define a polynomial $h$ on $\mathbb{R}^{n\times u\times m}$ by

$$h(A_1;A_2;\dots;A_m)=|H|.$$

We show that the polynomial $h(A_1;A_2;\dots;A_m)$ is not zero. It suffices to show that $h(A_1;A_2;\dots;A_m)\neq 0$ for some tensor $(A_1;A_2;\dots;A_m)$. We prepare a lemma.

$$\text{Let } f(a_1,\dots,a_{m-1},b)=\begin{vmatrix}a_1-b&\cdots&a_{m-1}-b\\a_1^2-b^2&\cdots&a_{m-1}^2-b^2\\\vdots&&\vdots\\a_1^{m-1}-b^{m-1}&\cdots&a_{m-1}^{m-1}-b^{m-1}\end{vmatrix}.$$

**Lemma 2.15** *If $a_1,\ldots,a_{m-1},b$ are distinct eath other, then* $f(a_1,\ldots,a_{m-1},b)\neq 0$.

**Proof** It is easy to see that

$$f(a_1,\ldots,a_m,b)=\begin{vmatrix} 1 & 0 & \cdots & 0 \\ b & a_1-b & \cdots & a_{m-1}-b \\ b^2 & a_1^2-b^2 & \cdots & a_{m-1}^2-b^2 \\ \vdots & \vdots & & \vdots \\ b^{m-1} & a_1^{m-1}-b^{m-1} & \cdots & a_{m-1}^{m-1}-b^{m-1} \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ b & a_1 & \cdots & a_{m-1} \\ b^2 & a_1^2 & \cdots & a_{m-1}^2 \\ \vdots & \vdots & & \vdots \\ b^{m-1} & a_1^{m-1} & \cdots & a_{m-1}^{m-1} \end{vmatrix}\neq 0.$$

∎

**Lemma 2.16** *Let $\boldsymbol{v}=(1,\ldots,1)^T\in\mathbb{R}^n$, $A_1=(E_n,\ldots,E_n,\boldsymbol{v},O_q)$ and*

$$A_{s+1}=(A_1\boldsymbol{e}_1,2^sA_1\boldsymbol{e}_2,\ldots,u^sA_1\boldsymbol{e}_u)=A_1\mathrm{Diag}(1^s,2^s,\ldots,u^s)$$

*for $1\leq s\leq m-1$. Then the $(u-1)\times u$ matrix $Y_j$ defined in* (2.13) *satisfies that $Y_j^{\perp}=t_j\boldsymbol{e}_j$ for some $t_j\neq 0$. In particular, $h(A_1;A_2;\ldots;A_m)\neq 0$.*

**Proof** Let

$$D_{t,s,j}=\mathrm{Diag}(((t-1)n+1)^s-j^s,((t-1)n+2)^s-j^s,\ldots,(tn)^s-j^s)$$

be an $n\times n$ matrix. Then

$$A_{s+1}-j^sA_1=(D_{1,s,j},D_{2,s,j},\ldots,D_{m-1,s,j},((p+1)^s-j^s)\boldsymbol{v},O_q).$$

For a $v\times w$ matrix $G=(g_{ij})$, we denote by

$$G^{=\{b_1,\ldots,b_r\}}_{=\{a_1,\ldots,a_c\}}$$

the $r\times c$ matrix obtained from $G$ by choosing $a_1$-, …, $a_c$-th columns and $b_1$-, …, $b_r$-th rows, that is $(g_{b_ia_j})$, and put

$$G_{=\{a_1,\ldots,a_c\}}=G^{=\{1,\ldots,v\}}_{=\{a_1,\ldots,a_c\}},\quad G_{\leq c}=G^{=\{1,\ldots,v\}}_{=\{1,\ldots,c\}},\quad G^{\leq r}_{\leq c}=G^{=\{1,\ldots,r\}}_{=\{1,\ldots,c\}}.$$

First we suppose that $j>p$. Put $S_t=\{t,n+t,2n+t,\ldots,(m-2)n+t\}$ and $M_{j,t}=(Y_j)^{=S_t}_{=S_t}=(X_j)^{=S_t}_{=S_t}$. Note that $M_{j,t}$ is nonsingular by Lemma 2.15, since

$$|M_{j,t}|=f(t,n+t,2n+t,\ldots,(m-2)n+t,j).$$

We consider the $p \times p$ matrix $(Y_j)^{\le p}_{\le p} = (X_j)_{\le p}$. There exists a permutation matrix $P$ such that

$$P^{-1}(X_j)_{\le p} P = \mathrm{Diag}(M_{j,1}, M_{j,2}, \ldots, M_{j,n}).$$

Thus we get

$$|(X_j)_{\le p}| = \prod_{1 \le t \le m-1} |M_{j,t}|$$

which implies that $(X_j)_{\le p}$ is nonsingular. Thus $\mathrm{rank} Y_j = u-1$ and $Y_j^{\perp} = t_j \boldsymbol{e}_j$ for some $t_j \ne 0$, since the $j$-th column vector of $Y_j$ is zero.

Next suppose that $j \le p$. The $j$-th column of $Y_j$ is zero. Let

$$Z_j = (X_j)_{=\{1,\ldots,p+1\} \setminus \{j\}}$$

be the $p \times p$ matrix obtain from $(X_j)_{\le p+1}$ by removing the $j$-th column. It suffices to show that $\mathrm{rank} Z_j = p$. We express $j$ uniquely by $ns_0 + t_0$ for a pair $(s_0, t_0)$ of integers with $0 \le s_0 \le m-2$ and $1 \le t_0 \le n$. Let

$$T = \{sn + t_0 \mid 0 \le s \le m-2, s \ne s_0\} \cup \{p+1\}.$$

There exist permutation matrices $P$ and $Q$ such that

$$PZ_jQ = \begin{pmatrix} \mathop{\mathrm{Diag}}\limits_{\substack{1 \le t \le n, \\ t \ne t_0}} M_{j,t} & O_{p-m+1,m-2} & * \\ O_{m-1,p-m+1} & & (X_j)^{=S_{t_0}}_{=T} \end{pmatrix}$$

of which last column corresponds to the $(p+1)$-th column of $X_j$. We get the equality

$$|Z_j| = (-1)^a |(X_j)^{=S_{t_0}}_{=T}| \prod_{1 \le t \le m-1, t \ne t_0} |M_{j,t}|.$$

Again by Lemma 2.15, $Z_j$ is nonsingular and $Y_j^{\perp} = t_j \boldsymbol{e}_j$ for some $t_j \ne 0$. ∎

Thus the polynomial $h$ is not zero. Consider a nonempty Zariski open set

$$S = \{(A_1; A_2; \ldots; A_m) \in \mathbb{R}^{n \times u \times m} \mid h(A_1; A_2; \ldots; A_m) \ne 0\}.$$

Note that the closure $\overline{S}$ of $S$ is equal to $\mathbb{R}^{n \times u \times m}$. For $(A_1; A_2; \ldots; A_m) \in S$ and $X_j, Y_j, H$ matrices given in (2.13) and (2.14), $A_k Y_j^{\perp} = j^{k-1} A_1 Y_j^{\perp}$ for $1 \le k \le m$ and $1 \le j \le u$. Since

$$\begin{aligned} A_k H &= (A_k Y_1^{\perp}, A_k Y_2^{\perp}, \ldots, A_k Y_u^{\perp}) \\ &= (A_1 Y_1^{\perp}, 2^{k-1} A_1 Y_2^{\perp}, \ldots, u^{k-1} A_1 Y_u^{\perp}) \\ &= A_1 H \mathrm{Diag}(1, 2^{k-1}, \ldots, u^{k-1}), \end{aligned}$$

it holds that $A_k = A_1 H \mathrm{Diag}(1, 2^{k-1}, \ldots, u^{k-1}) H^{-1}$ for each $k$. By Proposition 2.9, we get $\mathrm{rank}(A_1; A_2; \ldots; A_m) \le u$. Any number of $\mathrm{typical\_rank}_{\mathbb{R}}(m, u, n)$ is greater than or equal to $u$ which is equal to the generic rank of $\mathbb{C}^{m \times n \times u}$, since $(m-1)n < u < mn$. This completes the proof of Theorem 2.12.

**Corollary 2.17** *Let $3 \le m \le n$. Then the set of typical ranks of $m \times n \times (m-1)n$ tensors is either $\{(m-1)n\}$ or $\{(m-1)n, (m-1)n+1\}$.*

**Proof** The typical rank of $\mathbb{R}^{m\times n\times((m-1)n+1)}$ is only $(m-1)n+1$ by Theorem 2.12 and the minimal typical rank of $\mathbb{R}^{m\times n\times(m-1)n}$ is equal to $(m-1)n$, since it is equal to the generic rank of $\mathbb{C}^{m\times n\times(m-1)n}$. Thus the assertion follows from Proposition 2.5. ∎

# 3 Characterization

From now on, let $3 \le m \le n$, $\ell = m-1$ and $p = (m-1)n$. For an $n \times n \times \ell$ tensor $(Y_1;\dots;Y_\ell)$, consider an $n \times p \times m$ tensor $X(Y_1,\dots,Y_\ell) = (X_1;\dots;X_m)$ given by

$$\begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix} = \begin{pmatrix} E_n & & & \\ & E_n & & \\ & & \ddots & \\ & & & E_n \\ Y_1 & Y_2 & \cdots & Y_\ell \end{pmatrix}. \tag{3.1}$$

Note that $\operatorname{rank} X(Y_1,\dots,Y_\ell) \ge p$, since $\operatorname{rank} X(Y_1,\dots,Y_\ell)$ is greater than or equal to the rank of the $p \times p$ matrix (3.1). In generic, an $m \times n \times p$ tensor is equivalent to a tensor of type as $X(Y_1,\dots,Y_\ell)$.

We denote by $\mathfrak{M}$ the set of tensors $Y = (Y_1;\dots;Y_\ell) \in \mathbb{R}^{n\times n\times \ell}$ such that there exist an $m \times p$ matrix $(x_{ij})$ and an $n \times p$ matrix $A = (\boldsymbol{a}_1,\dots,\boldsymbol{a}_p)$ such that

$$(x_{1j}Y_1 + \cdots + x_{m-1,j}Y_{m-1} - x_{mj}E_n)\boldsymbol{a}_j = \mathbf{0} \tag{3.2}$$

for $1 \le j \le p$ and

$$B := \begin{pmatrix} AD_1 \\ \vdots \\ AD_\ell \end{pmatrix} \tag{3.3}$$

is nonsingular, where $D_k = \operatorname{Diag}(x_{k1},\cdots,x_{kp})$ for $1 \le k \le \ell$.

**Lemma 3.4** $\operatorname{rank} X(Y_1,\dots,Y_\ell) = p$ *if and only if* $(Y_1;\dots;Y_\ell) \in \mathfrak{M}$.

**Proof** Suppose that $\operatorname{rank} X(Y_1,\dots,Y_\ell) = p$. There are an $n \times p$ matrix $A$, a $p \times p$ matrix $Q$ and $p \times p$ diagonal matrices $D_i$ such that $X_k = AD_kQ$ for $k = 1,\dots,m$. Since

$$\begin{pmatrix} X_1 \\ \vdots \\ X_\ell \end{pmatrix} = E_p = \begin{pmatrix} AD_1 \\ \vdots \\ AD_\ell \end{pmatrix} Q,$$

$B$ is nonsingular. Then $(Y_1,\dots,Y_\ell)B = AD_m$ implies that $\sum_{k=1}^{\ell} Y_k AD_k = AD_m$. Therefore, the $j$-th column vector $\boldsymbol{a}_j$ of $A$ satisfies (3.2). Therefore $(Y_1,\dots,Y_\ell) \in \mathfrak{M}$. It is easy to see that the converse is also true. ∎

For an $n\times n\times \ell$ tensor $Y=(Y_1;\dots;Y_\ell)$, we put

$$V(Y)=\{\boldsymbol{a}\in\mathbb{R}^n \mid \sum_{k=1}^{\ell} x_kY_k\boldsymbol{a}=x_m\boldsymbol{a} \text{ for some } (x_1,\dots,x_m)^\top\neq \mathbf{0}\}.$$

The set $V(Y)$ is not a vector subspace of $\mathbb{R}^n$. Let $\hat{V}(Y)$ be the smallest vector subspace of $\mathbb{R}^n$ including $V(Y)$. Let

$$\mathfrak{S}=\{Y\in\mathbb{R}^{n\times n\times\ell} \mid \dim\hat{V}(Y)=n\}.$$

**Proposition 3.5** $\mathfrak{M}\subset\mathfrak{S}$ *holds.*

**Proof** Let $Y\in\mathfrak{M}$. Consider the matrix $B$ in (3.3) for any $m\times p$ matrix $(x_{ij})$ and any $n\times p$ matrix $A=(\boldsymbol{a}_1,\dots,\boldsymbol{a}_p)$ satisfying the equation (3.2). By column operations, $B$ is transformed to a $p\times p$ matrix having a form

$$\begin{pmatrix} P_{11} & O_{n,p-\dim\hat{V}(Y)} \\ P_{21} & P_{22}\end{pmatrix}$$

where $P_{11}$ is an $n\times\dim\hat{V}(Y)$ submatrix of $A$. Since $B$ is nonsingular, $P_{11}$ is also nonsingular, which implies that $\dim\hat{V}(Y)=n$. ∎

By Corollary 2.17, Lemma 3.4 and Proposition 3.5, we have the following

**Proposition 3.6** *If* $\operatorname{rank}X(Y)=p$ *then* $Y\in\mathfrak{S}$. *In particular,* $\overline{\mathfrak{S}}\neq\mathbb{R}^{n\times n\times\ell}$ *implies that* $\text{typical\_rank}_{\mathbb{R}}(m,n,p)=\{p,p+1\}$.

**Theorem 3.7 ([10])** *If* $(Y_1;\dots;Y_\ell;E_n)$ *is an absolutely nonsingular tensor, then it holds that* $\operatorname{rank}X(Y_1,\dots,Y_\ell)>p$.

Here $(Y_1;\dots;Y_\ell;Y_m)$ is called an absolutely nonsingular tensor if $|\sum_{k=1}^m x_kY_k|=0$ implies $(x_1,\dots,x_m)^\top=\mathbf{0}$. Therefore,

**Proposition 3.8** $\dim\hat{V}(Y)=0$ *if and only if* $(Y;E_n)$ *is an* $n\times n\times m$ *absolutely nonsingular tensor.*

Note that there exists an $n\times n\times m$ absolutely nonsingular tensor if and only if $m$ is less than or equal to the Hurwitz-Radon number $\rho(n)$ [10].

**Proposition 3.9** *Let* $Y$ *and* $Z$ *be* $n\times n\times m$ *tensors. Suppose* $(P,Q,R)\cdot Y=Z$ *for* $(P,Q,R)\in \mathrm{GL}(n)\times\mathrm{GL}(n)\times\mathrm{GL}(m)$. *Then* $V(Y)=Q^\top V(Z)=\{Q^\top\boldsymbol{y}\mid \boldsymbol{y}\in V(Z)\}$. *In particular,* $\dim\hat{V}(Z)=\dim\hat{V}(Y)$.

**Proof** Suppose that $\sum_{k=1}^m x_kZ_k\boldsymbol{y}=\mathbf{0}$. Then from the definition of the action, it follows that

$$\sum_{k=1}^m d_k\sum_{u=1}^m r_{ku}PY_uQ^\top\boldsymbol{y}=P(\sum_{u=1}^m(\sum_{k=1}^m d_kr_{ku}Y_u))Q^\top\boldsymbol{y}=\mathbf{0}.$$

Thus $Q^\top\boldsymbol{y}\in V(Y)$. ∎

**Corollary 3.10** $\mathfrak{S}$ *is closed under the equivalence relation.*

The closure of the set of all $n\times p\times m$ tensors equivalent to $X(Y_1,\dots,Y_\ell)$ for some $Y_1,\dots,Y_\ell$ is $\mathbb{R}^{n\times p\times m}$. Furthermore, the following claim holds. Let $\mathfrak{V}$ be the set of $n\times p\times m$ tensors $(X_1;\dots;X_m)$ such that $A=(X_1^\top,\dots,X_\ell^\top)$ is a nonsingular $p\times p$ matrix and $(Y_1;\dots;Y_\ell)$ given by $(Y_1,\dots,Y_\ell)=A^{-1}X_m$ lies in $\mathfrak{M}$. Any tensor of $\mathfrak{V}$ has rank $p$. If $\mathfrak{M}$ is dense in $\mathbb{R}^{n\times n\times\ell}$ then $\mathfrak{V}$ is dense in $\mathbb{R}^{n\times p\times m}$.

# 4 Classes of $n\times n\times\ell$ tensors

We separate $\mathbb{R}^{n\times n\times\ell}$ into three classes $\mathfrak{A}$, $\mathfrak{C}$, and $\mathfrak{B}$ as follows. Let $\mathfrak{A}$ be the set of tensors $Y$ such that $(Y;E_n)$ is absolutely nonsingular. By Proposition 3.8, we have the following

**Proposition 4.1** $\mathfrak{A}\cap\mathfrak{S}=\varnothing$.

From now on, we use symbols $x_1,\dots,x_\ell,x_m$ as indeterminates over $\mathbb{R}$. For $Y=(Y_1;\dots;Y_\ell)\in\mathbb{R}^{n\times n\times\ell}$, we define the $n\times n$ matrix with entries in $\mathbb{R}[x_1,\dots,x_\ell,x_m]$ as follows.

$$M(\boldsymbol{x},Y)=\sum_{k=1}^{\ell}x_kY_k-x_mE_n$$

Note that fixing $a_1,\dots,a_\ell$, the determinant $|M(\boldsymbol{a},Y)|$ is positive for $a_m\ll 0$, where $\boldsymbol{a}=(a_1,\dots,a_\ell,a_m)^\top$. Set

$$\mathfrak{C}=\{Y\in\mathbb{R}^{n\times n\times\ell}\mid |M(\boldsymbol{a},Y)|<0 \text{ for some } \boldsymbol{a}\in\mathbb{R}^m\}.$$

Note that $\mathfrak{C}$ is not empty, and if $n$ is not congruent to 0 modulo 4 then $\mathfrak{A}$ is empty since $m\geq 3$. Set $\mathfrak{B}=\mathbb{R}^{n\times n\times\ell}\setminus(\mathfrak{A}\cup\mathfrak{C})$. The class $\mathfrak{B}$ contains the zero tensor.

**Proposition 4.2** $\mathfrak{A}$ *and* $\mathfrak{C}$ *are open subsets of* $\mathbb{R}^{n\times n\times\ell}$.

Recall that

$$\mathfrak{A}=\{Y\in\mathbb{R}^{n\times n\times\ell}\mid |M(\boldsymbol{a},Y)|>0 \text{ for all } \boldsymbol{a}\neq\boldsymbol{0}\}.$$

Thus it holds

$$\mathfrak{B}=\{Y\in\mathbb{R}^{n\times n\times\ell}\mid \begin{array}{l}|M(\boldsymbol{b},Y)|=0 \text{ for some } \boldsymbol{b}\neq\boldsymbol{0} \text{ and}\\ |M(\boldsymbol{a},Y)|\geq 0 \text{ for all } \boldsymbol{a}\end{array}\}.$$

**Proposition 4.3** $\mathfrak{B}$ *is a boundary of* $\mathfrak{C}$. *In particular,* $\mathbb{R}^{n\times n\times\ell}$ *is a disjoint sum of* $\mathfrak{A}$ *and the closure* $\overline{\mathfrak{C}}$ *of* $\mathfrak{C}$.

**Proof** It suffices to show that $\mathfrak{B}\subset\overline{\mathfrak{C}}$. Let $Y=(Y_1;\dots;Y_\ell)\in\mathfrak{B}$. There are a nonzero vector $\boldsymbol{b}=(b_1,\dots,b_\ell,b_m)^\top\in\mathbb{R}^n$ with $|M(\boldsymbol{b},Y)|=0$ and an element $g\in\mathrm{GL}(\ell)$ such that $g\cdot Y=(Z_1;Z_2;\dots;Z_\ell)$ and $Z_1=\sum_{k=1}^{\ell}b_kY_k$. Then $|Z_1-b_mE_n|=0$. Take a sequence $\{Z_1^{(u)}\}_{u\geq 1}$ such that $|Z_1^{(u)}-b_mE_n|<0$ and $\lim_{u\to\infty}Z_1^{(u)}=Z_1$. Thus, $(Z_1^{(u)};Z_2;\dots;Z_\ell)\in\mathfrak{C}$ and then $g^{-1}\cdot(Z_1^{(u)};Z_2;\dots;Z_\ell)\in\mathfrak{C}$. Therefore, $Y\in\overline{\mathfrak{C}}$. ∎

**Corollary 4.4** *If $\mathfrak{A}$ is not empty then $\mathfrak{B}$ is a boundary of $\mathfrak{A}$.*

The set $\mathfrak{B}$ contains a nonzero tensor in general. We give an example.

**Example 4.5** Let $A = (A_1;A_2;A_3)$ be a $6\times 6\times 3$ tensor given by

$$X(x_1,x_2,x_3) = x_1A_1 + x_2A_2 - x_3A_3 = \begin{pmatrix} -x_3 & -x_2 & 0 & 0 & 0 & -x_1 \\ x_1 & -x_3 & x_2 & 0 & 0 & 0 \\ 0 & x_1 & -x_3 & x_2 & 0 & 0 \\ 0 & 0 & x_1 & -x_3 & -x_2 & 0 \\ 0 & 0 & 0 & x_1 & -x_3 & x_2 \\ -x_2 & 0 & 0 & 0 & x_1 & -x_3 \end{pmatrix}$$

Then $|a_1A_1+a_2A_2-a_3A_3| = a_3^2(a_1a_2-a_3^2)^2+(a_1^3+a_2^3)^2 \geq 0$. The equality holds if $a_3=0$ and $a_1=-a_2$. Thus $\dim\hat{V}((A_1;A_2))=1$. Let $B = \begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ be a $6\times 6$ matrix. If $x_3 = y$, $x_1 = -y^2$, and $x_2 = -2y/5$, then

$$|X+yB| = y^6(y^6+y^5-7y^4/5+161y^3/125-167y^2/125+629y/625-2926/15625).$$

Thus, if $|a_3|$ is sufficiently small then $|X(-a_3^2,-2a_3/5,a_3)+a_3B|<0$.

**Proposition 4.6** *If $m \leq \rho(n-1)$ then $\mathfrak{C} \not\subset \mathfrak{S}$, where $\rho(n-1)$ is a Hurwitz-Radon number.*

**Proof** Let $(A_1;\ldots;A_\ell;E_{n-1})$ be an $(n-1)\times(n-1)\times m$ absolutely nonsingular tensor. Put $B_k = \mathrm{Diag}(a_k,A_k)$ for $1\leq k\leq \ell$ and $B_m = \mathrm{Diag}(1,E_{n-1}) = E_n$, and $B=(B_1;\ldots;B_\ell)$. Then it is easy to see that $B\in\mathfrak{C}$ and $|\sum_{k=1}^{\ell} x_kB_k - zB_m| = 0$ implies $z = \sum_{k=1}^{\ell} a_kx_k$. Therefore $V(B) = \{a(1,0,\ldots,0)^\top \in \mathbb{R}^n \mid a\in\mathbb{R}\}$. In particular $B\notin\mathfrak{S}$. ∎

# 5 Irreducibility

In the space of homogeneous polynomials in $m$ variables, there exists a proper Zariski closed subset $S$ such that if a polynomial does not belong to $S$ then it is irreducible [5, Theorem 7], since $m\geq 3$. Let $P(m,n)$ be the set of homogeneous polynomials in $m$ variables $x_1,\ldots,x_m$ with real coefficients of degree $n$ such that the coefficient of $x_m^n$ is one. Its dimension is $\binom{m+n-1}{m-1}-1$. Let $I_\ell$ be a nonempty Zariski open subset of $P(m,n)$ such that any polynomial of $I_\ell$ is irreducible. Note that $|-M(\boldsymbol{x},Y)|\in P(m,n)$. This section stands to show the following fact.

**Proposition 5.1** *The set*

$$\{Y\in\mathbb{R}^{n\times n\times\ell} \mid |-M(\boldsymbol{x},Y)|\in I_\ell\}$$

*is a nonempty Zariski open subset of $\mathbb{R}^{n\times n\times\ell}$.*

Let $f_\ell \colon \mathbb{R}^{n\times n\times \ell} \to P(m,n)$ be a map which sends $(Y_1;\dots;Y_\ell)$ to $|\sum_{k=1}^{\ell} x_k Y_k + x_m E_n|$. Note that $|-M(\boldsymbol{x},Y)| \in I_\ell$ if and only if $f_\ell(Y) \in I_\ell$. Since $I_\ell$ is a Zariski open set,

$$\mathfrak{T}_\ell := \{Y \in \mathbb{R}^{n\times n\times \ell} \mid f_\ell(Y) \in I_\ell\}$$

is a Zariski open subset of $\mathbb{R}^{n\times n\times \ell}$. Then it suffices to show that $\mathfrak{T}_\ell$ is not empty. First, we show it in the case where $m=3$.

The affine space $P(3,n)$ is isomorphic to a real vector space of dimension $n(n+3)/2$ with basis

$$\{x_1^a x_2^b x_3^c \mid 0 \le a,b,c \le n, a+b+c=n, c \ne n\}.$$

Let $G$ be a map from $\mathbb{R}^{n\times n\times 2}$ to $\mathbb{R}^{n(n+3)/2}$ defined as

$$G((Y_1;Y_2)) = \phi(|x_1Y_1 + x_2Y_2 + x_3E_n|),$$

where $\phi \colon P(3,n) \to \mathbb{R}^{n(n+3)/2}$ is an isomorphism. It suffices to show that the Jacobian matrix of $G$ has generically full column rank. To show this, we restrict the source of $G$ to

$$S := \{(Y_1;Y_2) \in \mathbb{R}^{n\times n\times 2} \mid Y_1 = \begin{pmatrix} u_{11} & 0 & \cdots & 0 \\ u_{21} & u_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{n1} & \cdots & u_{n-1,1} & u_{n1} \end{pmatrix}, Y_2 = \begin{pmatrix} 0 & 0 & \cdots & v_1 \\ -1 & 0 & \cdots & v_2 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & v_n \end{pmatrix}\}$$

of dimension $n(n+3)/2$, say $G|_S \colon S \to \mathbb{R}^{n(n+3)/2}$.

**Lemma 5.2** *The Jacobian of $G|_S$ is nonzero.*

**Proof** Put $g(Y) := f(Y) - x_3^n$ for $Y \in S$. Suppose that for constants $c(v_j)$, $c(u_{ij})$, the linear equation

$$\sum_{j=1}^{n} c(v_j)\frac{\partial g}{\partial v_j} + \sum_{1\le j\le i\le n} c(u_{ij})\frac{\partial g}{\partial u_{ij}} = 0 \tag{5.3}$$

holds. We show that all of $c(v_j)$, $c(u_{ij})$ are zero by induction on $n$. It is easy to see that the assertion holds in the case where $n=1$. As the induction assumption, we assume that the assertion holds in the case where $n-1$ instead of $n$. We put

$$\lambda_j = u_{jj}x_1 + x_3 \text{ and } \mu(a,b) = \prod_{t=a}^{b} \lambda_t.$$

After a partial derivation, we put $u_{ij} = 0$ $(i > j)$ and then have the following equations:

$$\begin{aligned}
\frac{\partial g}{\partial v_j} &= x_2^{n-j+1}\mu(1,j-1) && (1\le j\le n) \\
\frac{\partial g}{\partial u_{jj}} &= x_1\mu(1,j-1)\begin{vmatrix} \lambda_{j+1} & & & & v_{j+1}x_2 \\ -x_2 & \lambda_{j+2} & & & v_{j+2}x_2 \\ & \ddots & \ddots & & \vdots \\ & & -x_2 & \lambda_{n-1} & v_{n-1}x_2 \\ & & & -x_2 & \lambda_n+v_nx_2 \end{vmatrix} && (1\le j\le n) \\
\frac{\partial g}{\partial u_{ij}} &= -x_1x_2^{n-i}\mu(j+1,i-1)\begin{vmatrix} \lambda_1 & & & & v_1x_2 \\ -x_2 & \lambda_2 & & & v_2x_2 \\ & \ddots & \ddots & & \vdots \\ & & -x_2 & \lambda_{j-1} & v_{j-1}x_2 \\ & & & -x_2 & v_jx_2 \end{vmatrix} && (1\le j<i\le n)
\end{aligned}$$

By seeing terms divisible by $\lambda_1$ in the left hand side of (5.3), we have

$$\sum_{j=2}^{n} c(v_j)\frac{\partial g}{\partial v_j}+\sum_{2\le j\le i\le n} c(u_{ij})h_{ij}=0,$$

where

$$h_{ij}=-x_1x_2^{n-i}\mu(j+1,i-1))\begin{vmatrix} \lambda_1 & & & & 0 \\ -x_2 & \lambda_2 & & & v_2x_2 \\ & \ddots & \ddots & & \vdots \\ & & -x_2 & \lambda_{j-1} & v_{j-1}x_2 \\ & & & -x_2 & v_jx_2 \end{vmatrix}.$$

Note that

$$\begin{aligned}
\frac{\partial g}{\partial v_j} &= \lambda_1\frac{\partial g'}{\partial v_j} \quad (2\le j\le n), \text{ and} \\
h_{ij} &= \lambda_1\frac{\partial g'}{\partial u_{ij}} \quad (2\le j\le i\le n)
\end{aligned}$$

where $g'$ is the determinant of the $(n-1)\times(n-1)$ matrix obtained from $x_1Y_1+x_2Y_2+x_3E_n$ by removing the first row and the first column minus $x_3^{n-1}$. Therefore by the induction assumption,

$$c(v_j)=c(u_{ij})=0 \quad (2\le j\le i\le n)$$

since $\frac{\partial g'}{\partial v_j}, \frac{\partial g'}{\partial u_{ij}}$ $(2\le j\le i\le n)$ are linearly independent. By (5.3), we have

$$c(v_1)x_2^n+c(u_{11})\frac{\partial g}{\partial u_{11}}-\sum_{i=2}^{n} c(u_{i1})v_1x_1x_2^{n-i+1}\mu(2,i-1)=0. \tag{5.4}$$

By expanding at the $n$-th column, we have

$$\frac{\partial g}{\partial u_{11}}=\sum_{i=2}^{n-1} v_ix_1x_2^{n-i-1}\mu(2,i-1)+x_1(\lambda_n+v_nx_2)\mu(2,n-1).$$

Therefore, the equation (5.4) implies that

$$c(v_1)x_2^n + \sum_{i=2}^{n}(c(u_{11})v_i - c(u_{i1})v_1)x_1x_2^{n-i+1}\mu(2,i-1) + c(u_{11})x_1\mu(2,n) = 0.$$

In this equation we notice the coefficients corresponding to $x_2^s$, $0 \le s \le n$. Then we have $c(u_{i1}) = c(v_1) = 0$ for $1 \le i \le n$.

Therefore, we conclude that $\dfrac{\partial g}{\partial v_j}, \dfrac{\partial g}{\partial u_{ij}}$ $(1 \le j \le i \le n)$ are linearly independent, which means that the Jacobian of $G|_S$ is nonzero. ∎

By Lemma 5.2, there is an open subset $S$ of $\mathbb{R}^{n\times n\times 2}$ such that the rank of the Jacobian matrix of $G$ at $Y$ has full column rank for any $Y \in S$. Then $f_2(S)\cap I_2$ is not empty and thus $\mathfrak{T}_2 \cap S$ is not empty. In particular, $\mathfrak{T}_2$ is not empty.

Now we show that $\mathfrak{T}_\ell$ is not empty in the case where $\ell > 2$. Let $q\colon \mathbb{R}^{n\times n\times \ell} \to \mathbb{R}^{n\times n\times 2}$ be a canonical projection which sends $(Y_1;\ldots;Y_\ell)$ to $(Y_{\ell-1};Y_\ell)$. Put $\hat{\mathfrak{T}} = q^{-1}(\mathfrak{T}_2\cap S)$ and let $\bar{q}\colon P(m,n) \to P(3,n)$ be also a canonical projection which sends a polynomial $g(x_1,\ldots,x_m)$ to $g(0,\ldots,0,x_1,x_2,x_3)$. The following diagram is commutative.

$$\begin{array}{ccccc}
\hat{\mathfrak{T}} & \xrightarrow{\subset} & \mathbb{R}^{n\times n\times\ell} & \xrightarrow{f_\ell} & P(m,n) \\
\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle \bar{q}}\downarrow \\
\mathfrak{T}_2\cap S & \xrightarrow{\subset} & \mathbb{R}^{n\times n\times 2} & \xrightarrow{f_2} & P(3,n)
\end{array}$$

Note that if $g(x_1,\ldots,x_m) \in P(m,n)$ is reducible then so is $g(0,\ldots,0,x_1,x_2,x_3) \in P(3,n)$. The set $\hat{\mathfrak{T}}$ is a nonempty open subset of $\mathbb{R}^{n\times n\times\ell}$ with the property that $f_\ell(Y)$ is irreducible for any $Y \in \hat{\mathfrak{T}}$. Thus $\mathfrak{T}_\ell$ is not empty, since $\hat{\mathfrak{T}} \subset \mathfrak{T}_\ell$. This completes the proof of Proposition 5.1.

# 6 Proof of Theorem 1.1

In this section we show Theorem 1.1.

Let $\check{\boldsymbol{x}} = (x_1,\ldots,x_\ell)^\top$ for $\boldsymbol{x} = (x_1,\ldots,x_\ell,x_m)^\top$, and put

$$\psi(\boldsymbol{x},Y) := \begin{pmatrix} (-1)^{n+1}|M(\boldsymbol{x},Y)_{n,1}| \\ (-1)^{n+2}|M(\boldsymbol{x},Y)_{n,2}| \\ \vdots \\ (-1)^{n+n}|M(\boldsymbol{x},Y)_{n,n}| \end{pmatrix}, \quad \check{\boldsymbol{x}} \otimes \psi(\boldsymbol{x},Y) := \begin{pmatrix} x_1\psi(\boldsymbol{x},Y) \\ x_2\psi(\boldsymbol{x},Y) \\ \vdots \\ x_\ell\psi(\boldsymbol{x},Y) \end{pmatrix}$$

and

$$U(Y) := \langle \check{\boldsymbol{a}} \otimes \psi(\boldsymbol{a},Y) \mid |M(\boldsymbol{a},Y)| = 0 \rangle.$$

**Lemma 6.1** *If* $\dim U(Y) = p$*, then* $Y \in \mathfrak{M}$*.*

**Proof** Let $\dim U(Y) = p$. Then there are $\boldsymbol{a}_j = (a_{1j}, \ldots, a_{mj})^\top \in U(Y)$ for $1 \leq j \leq p$ such that

$$B' = (\check{\boldsymbol{a}}_1 \otimes \psi(\boldsymbol{a}_1, Y), \ldots, \check{\boldsymbol{a}}_p \otimes \psi(\boldsymbol{a}_p, Y))$$

is nonsingular. Note that $M(\boldsymbol{a}_j, Y)\psi(\boldsymbol{a}_j, Y) = \mathbf{0}$ for $1 \leq j \leq p$ and

$$B' = \begin{pmatrix} AD_1 \\ \vdots \\ AD_\ell \end{pmatrix},$$

where $A = (\psi(\boldsymbol{a}_1, Y), \ldots, \psi(\boldsymbol{a}_p, Y))$ and $D_k = \mathrm{Diag}(a_{k1}, \cdots, a_{kp})$ for $1 \leq k \leq \ell$. Thus $Y \in \mathfrak{M}$. ∎

For an $n \times \ell$ matrix $C = (\boldsymbol{c}_1, \ldots \boldsymbol{c}_\ell)$, we put

$$g(\boldsymbol{x}, Y, C) := \begin{vmatrix} M(\boldsymbol{x}, Y)^{<n} \\ \sum_{k=1}^{\ell} x_k \boldsymbol{c}_k^\top \end{vmatrix},$$

where $M(\boldsymbol{x}, Y)^{<n}$ is the $(n-1) \times n$ matrix obtained from $M(\boldsymbol{x}, Y)$ by removing the $n$-th row.

**Lemma 6.2** *Let $C = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_\ell)$ be an $n \times \ell$ matrix. The following claims are equivalent.*

(1) $\dim U(Y) = p$.

(2) $g(\boldsymbol{a}, Y, C) = 0$ *for any* $\boldsymbol{a} \in \mathbb{R}^m$ *with* $|M(\boldsymbol{a}, Y)| = 0$ *implies* $C = O$.

**Proof** Let $C = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_\ell)$ be an $n \times \ell$ matrix. Put $\boldsymbol{d} = (\boldsymbol{c}_1^\top, \ldots, \boldsymbol{c}_\ell^\top)^\top \in \mathbb{R}^p$. The inner product of this vector $\boldsymbol{d}$ with $\check{\boldsymbol{a}} \otimes \psi(\boldsymbol{a}, Y)$ is equal to $g(\boldsymbol{a}, Y, C)$. Therefore $\boldsymbol{d}$ belongs to the orthogonal complement of $U(Y)$ if and only if $g(\boldsymbol{x}, Y, C) = 0$ for any $\boldsymbol{a} \in \mathbb{R}^m$ with $|M(\boldsymbol{a}, Y)| = 0$. Thus the assertion holds. ∎

For any $i$ and $k$ with $1 \leq i \leq n-1$ and $1 \leq k \leq n$, let $s_i^{(k)}$ be an elementary symmetric polynomial of degree $i$ with variables $\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n$. Put

$$S_n = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ s_1^{(1)} & s_1^{(2)} & \ldots & s_1^{(n)} \\ s_2^{(1)} & s_2^{(2)} & \ldots & s_2^{(n)} \\ \vdots & \vdots & & \vdots \\ s_{n-1}^{(1)} & s_{n-1}^{(2)} & \ldots & s_{n-1}^{(n)} \end{pmatrix}.$$

**Lemma 6.3** *The determinant $|S_n|$ of the $n \times n$ matrix $S_n$ is equal to*

$$\prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j).$$

*In particular, if $\alpha_1, \ldots, \alpha_n$ are distinct each other, then $S_n$ is nonsingular.*

**Proof** For any $i$ and $k$ with $1 \leq i \leq n-1$ and $2 \leq k \leq n-1$, let $t_i^{(k-1)}$ be an elementary symmetric polynomial of degree $i$ with variables $\alpha_2, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n$. For $1 \leq i \leq n-1$ and $1 \leq k \leq n$, we have $s_i^{(k)} - s_i^{(1)} = (\alpha_1 - \alpha_k) t_{i-1}^{(k-1)}$. Then

$$|S_n| = \prod_{2 \leq k \leq n} (\alpha_1 - \alpha_k) \begin{vmatrix} 1 & 1 & \ldots & 1 \\ t_1^{(1)} & t_1^{(2)} & \ldots & t_1^{(n-1)} \\ \vdots & \vdots & & \vdots \\ t_{n-2}^{(1)} & t_{n-2}^{(2)} & \ldots & t_{n-2}^{(n-1)} \end{vmatrix}.$$

Therefore we have the assertion by induction on $n$. ∎

The following lemma is obtained straightforwardly.

**Lemma 6.4**

$$\begin{vmatrix} \alpha_1 + z & & & & a_1 \\ & \alpha_2 + z & & & a_2 \\ & & \ddots & & \vdots \\ & & & \alpha_n + z & a_n \\ b_1 & b_2 & \ldots & b_n & 0 \end{vmatrix} = -(z^{n-1}, z^{n-2}, \ldots, 1) S_n \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \\ \vdots \\ a_n b_n \end{pmatrix}.$$

**Proof** We see the left hand of the equation is equal to

$$\begin{aligned} & -\sum_{k=1}^{n} a_k b_k \frac{\prod_{1 \leq i \leq n} (\alpha_i + z)}{\alpha_k + z} \\ &= -\sum_{k=1}^{n} a_k b_k \left( \sum_{i=1}^{n} s_{i-1}^{(k)} \right) z^{n-i} \\ &= -\sum_{i=1}^{n} \left( \sum_{k=1}^{n} a_k b_k s_{i-1}^{(k)} \right) z^{n-i} \\ &= -(z^{n-1}, z^{n-2}, \ldots, 1) \begin{pmatrix} \sum_{k=1}^{n} a_k b_k \\ \sum_{k=1}^{n} a_k b_k s_1^{(k)} \\ \vdots \\ \sum_{k=1}^{n} a_k b_k s_{n-1}^{(k)} \end{pmatrix}. \end{aligned}$$

∎

**Corollary 6.5** *Let $\alpha_1, \ldots, \alpha_{n-1}$ be distinct complex numbers, $a_1, \ldots, a_{n-1}$ nonzero complex numbers, and $b_1, \ldots, b_{n-1}$ complex numbers. If*

$$\begin{vmatrix} \mathrm{Diag}(\alpha_1, \ldots, \alpha_{n-1}) + zE_{n-1} & \boldsymbol{a} \\ \boldsymbol{b}^\top & 0 \end{vmatrix} = 0$$

*for any $z \in \mathbb{R}$, then $\boldsymbol{b} = \boldsymbol{0}$, where $\boldsymbol{a} = (a_1 \ldots, a_{n-1})^\top$ and $\boldsymbol{b} = (b_1, \ldots, b_{n-1})^\top$.*

**Proof** Since $S_n \begin{pmatrix} a_1b_1 \\ a_2b_2 \\ \vdots \\ a_nb_n \end{pmatrix} = \mathbf{0}$ and $S_n$ is nonsingular, we have $(a_1b_1,\ldots,a_nb_n) = \mathbf{0}^\top$. ∎

The set

$$\mathfrak{U}_1 = \{Y \in \mathbb{R}^{n\times n\times \ell} \mid |M(\boldsymbol{x},Y)| \text{ is irreducible}\}$$

is a nonempty Zariski open subset of $\mathbb{R}^{n\times n\times \ell}$ (see Proposition 5.1). Let $W$ be the subset of $\mathbb{R}^{n\times n}$ consisting of matrices $\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ such that all eigenvalues of $A_1$ are distinct over the complex number field and every element of the vector $P^{-1}A_2$ is nonzero complex number where $A_1 \in \mathbb{R}^{(n-1)\times(n-1)}$, $P \in \mathbb{C}^{(n-1)\times(n-1)}$ with $P^{-1}A_1P$ is a diagonal matrix. Note that the validity of the condition that every element of the vector $P^{-1}A_2$ is nonzero is independent of the choice of $P$. We put

$$\mathfrak{U}_2 := \{(Y_1;\ldots;Y_\ell) \in \mathbb{R}^{n\times n\times \ell} \mid Y_k \in W, 1 \le k \le \ell\}.$$

The set $\mathfrak{U}_2$ is a nonempty Zariski open subset of $\mathbb{R}^{n\times n\times \ell}$ and $\mathfrak{U} := \mathfrak{U}_1 \cap \mathfrak{U}_2$ is also.

**Lemma 6.6** *Let $Y \in \mathfrak{U}_2$ and $\boldsymbol{d}_1,\ldots,\boldsymbol{d}_\ell \in \mathbb{R}^{n-1}$. If*

$$\begin{vmatrix} M(\boldsymbol{a},Y)^{<n} & \\ \sum_{k=1}^{\ell} a_k\boldsymbol{d}_k^\top & 0 \end{vmatrix} = 0$$

*for any $\boldsymbol{a} = (a_1,\ldots,a_m)^\top \in \mathbb{R}^m$, then $\boldsymbol{d}_1 = \cdots = \boldsymbol{d}_\ell = \mathbf{0}$.*

**Proof** Let $1 \le k \le \ell$. Take $a_k = 1$ and $a_j = 0$ for $1 \le j \le \ell$, $j \ne k$ and put $Y_k = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, where $A_1$ is an $(n-1)\times(n-1)$ matrix. Since $Y_k \in W$, there are a matrix $P \in \mathbb{C}^{(n-1)\times(n-1)}$ and distinct complex numbers $\alpha_1,\ldots,\alpha_{n-1}$ such that

$$\mathrm{Diag}(P,1)^{-1}\begin{pmatrix} (Y_k - a_mE_n)^{<n} & \\ \boldsymbol{d}_k^\top & 0 \end{pmatrix}\mathrm{Diag}(P,1) = \begin{pmatrix} \mathrm{Diag}(\alpha_1,\ldots,\alpha_{n-1}) - a_mE_{n-1} & P^{-1}A_2 \\ \boldsymbol{d}_k^\top P & 0 \end{pmatrix}$$

and every element of $P^{-1}A_2$ is nonzero. Then we have $\boldsymbol{d}_k^\top P = \mathbf{0}^\top$ by Corollary 6.5 and thus $\boldsymbol{d}_k = \mathbf{0}$. ∎

The following lemma is essential for the proof of Theorem 1.1.

**Lemma 6.7** $\mathfrak{U} \cap \mathfrak{C} \subset \mathfrak{M}$. *In particular, $\overline{\mathfrak{C}} \subset \overline{\mathfrak{M}}$ holds.*

**Proof** Let $Y \in \mathfrak{U} \cap \mathfrak{C}$ and fix it. There exists $\boldsymbol{a} = (a_1,\ldots,a_\ell,a_m)^\top$ such that $|M(\boldsymbol{a},Y)| < 0$. Then there is an open neighborhood $U$ of $(a_1,\ldots,a_\ell)^\top$ and a mapping $\mu : U \to \mathbb{R}$ such that

$$|M(\begin{pmatrix} \boldsymbol{y} \\ \mu(\boldsymbol{y}) \end{pmatrix}, Y)| = 0$$

for any $\boldsymbol{y} \in U$. Thus $|M(\boldsymbol{x},Y)| = 0$ determines an $(m-1)$-dimensional algebraic set. Let $C$ be an $n \times \ell$ matrix. Now suppose that $g(\boldsymbol{a},Y,C) = 0$ holds for any $\boldsymbol{a} \in \mathbb{R}^m$ with $|M(\boldsymbol{a},Y)| = 0$. We show that $g(\boldsymbol{x},Y,C)$ is zero as a polynomial over elements of $\boldsymbol{x}$. As a contrary, assume that $g(\boldsymbol{x},Y,C)$ is not zero. The degree of $g(\boldsymbol{x},Y,C)$ corresponding to the $m$-th element of $\boldsymbol{x}$ is less than $m$ which is that of $|M(\boldsymbol{x},Y)|$. Furthermore, since $M(\boldsymbol{x},Y)$ is irreducible, $M(\boldsymbol{x},Y)$ and $g(\boldsymbol{x},Y,C)$ are coprime. Then there are polynomials $f_1(\boldsymbol{x})$, $f_2(\boldsymbol{x}) \in \mathbb{R}[x_1,\ldots,x_\ell,x_m]$ and a nonzero polynomial $h(\check{\boldsymbol{x}}) \in \mathbb{R}[x_1,\ldots,x_\ell]$ such that

$$f_1(\boldsymbol{x})M(\boldsymbol{x},Y) + f_2(\boldsymbol{x})g(\boldsymbol{x},Y,C) = h(\check{\boldsymbol{x}})$$

as a polynomial over elements of $\boldsymbol{x}$, by Euclidean algorithm. However, we can take $\boldsymbol{b} \in U$ so that $h(\boldsymbol{b}) \neq 0$. Then the above equation does not hold at $\boldsymbol{x} = \begin{pmatrix} \boldsymbol{b} \\ \mu(\boldsymbol{b}) \end{pmatrix}$. Hence $g(\boldsymbol{x},Y,C)$ must be the zero polynomial over elements of $\boldsymbol{x}$. Let $\boldsymbol{c}_k^\top = (c_{1k},\ldots,c_{nk})$. By seeing the coefficient of $x_m^{n-1}x_k$, we get $c_{nk} = 0$ for $1 \leq k \leq \ell$. Therefore $C = O$ by Lemma 6.6. By Lemmas 6.2 and 6.1 we get $Y \in \mathfrak{M}$. Therefore $\mathfrak{U} \cap \mathfrak{C}$ is a subset of $\mathfrak{M}$. Then $\overline{\mathfrak{C}} = \overline{\mathfrak{U} \cap \mathfrak{C}} \subset \overline{\mathfrak{M}}$. ∎

**Theorem 6.8** $\overline{\mathfrak{S}} = \overline{\mathfrak{M}} = \overline{\mathfrak{C}}$ *holds.*

**Proof** We have $\overline{\mathfrak{M}} \subset \overline{\mathfrak{S}}$ by Proposition 3.5. By Propositions 4.1 and 4.3, the set $\mathfrak{S}$ is a subset of $\overline{\mathfrak{C}}$ and then $\overline{\mathfrak{S}} \subset \overline{\mathfrak{C}}$. Therefore $\overline{\mathfrak{S}} = \overline{\mathfrak{M}} = \overline{\mathfrak{C}}$ by Lemma 6.7. ∎

**Proof of Theorem 1.1.** For almost all $Y \in \mathfrak{A}$, $\operatorname{rank} X(Y) = p+1$ by Theorem 3.7. Since $\mathfrak{A}$ is an open set, if $\mathfrak{A}$ is not an empty set, then $\text{typical\_rank}_{\mathbb{R}}(m,n,p) = \{p, p+1\}$ ([10, Theorem 3.4]). Suppose that $\mathfrak{A}$ is empty. Then $\overline{\mathfrak{M}} = \mathbb{R}^{n\times n\times \ell}$ and the closure of the set consisting of all $n \times p \times m$ tensors equivalent to $X(Y)$ for some $Y \in \mathfrak{M}$ is $\mathbb{R}^{n\times p\times m}$. Recall that any tensor $X(Y)$ for $Y \in \mathfrak{M}$ has rank $p$. By Theorem 2.2, $p$ is the maximal typical rank of $\mathbb{R}^{n\times p\times m}$. Therefore,

$$\text{typical\_rank}_{\mathbb{R}}(m,n,p) = \text{typical\_rank}_{\mathbb{R}}(n,p,m) = \{p\}$$

holds. ∎

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